Pergamon

PII: S0021-8928(96)00111-6

J. Appl. Maths Mechs, Vol. 60, No. 6, pp. 891–897, 1996

0021-8928/96 $24.00+0.00

ON THE NEED TO TAKE INTO ACCOUNT THE POTENTIAL ENERGY OF MASS SYSTEMS†

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(*Received* 20 *June* 1996)

A gravitation theory is constructed taking the thermodynamic and potential energy into account and also the phenomenon of weightlessness, using the presence of a potential energy, which is additionally justified theoretically and experimentally. A theory of the motion of gravitating masses in the Special Theory of Relativity is proposed for absolute and relative motion using proper times for the individual masses. This theory touches directly on Newtonian theory, but differs from it and from the General Theory of Relativity (in particular, when the rotation of the masses and also weightlessness are taken into account). A gravitation theory that is covariant in the Fermi variables and invariant under Lorentz transformations is constructed.

The phenomenon of weightlessness, observed when material bodies move in space, is due to the presence of masses and their mutual attractive forces, proportional to the masses, and the geometrical properties of spaces as carriers of the corresponding mechanical and physical events.

Model laws are constructed in the gravitation theory for local and global interactions between moving masses, and, on the basis of these, mathematical methods are derived for describing the free motions of systems of material bodies and their components. Interaction occurs due to the action of the dynamic force of gravity, described, on the basis of observations, in the form $d\mathbf{P} = dm \cdot \mathbf{g}$ and due to attraction between all the freely moving elementary masses in the system of material bodies considered.

The attractive forces are balanced by inertial forces, which are also proportional to the moving masses, and they can also be interpreted as reactive forces, generated by the kinetic properties of spaces. They are expressed as a function of the corresponding accelerations $\mathbf{a}$ of the mass elements for the inertial force by formulae of the form $dm.\mathbf{a}$ and by postulating the locally correct equation

$$d\mathbf{P} - dm \cdot \mathbf{a} = 0, \text{ or } \mathbf{g} = \mathbf{a} \tag{1}$$

when the invariant vectors $\mathbf{g}$ and $\mathbf{a}$ may be non-zero. In particular, in classical models of gravitation theory, it is assumed that there are no other interaction forces, in particular, contact interactions.

In Newtonian theory, the important inequality $\mathbf{g} \neq 0$ is taken into account, while in the General Theory of Relativity, instead of the force of gravity $\mathbf{P}$, a curved four-dimensional pseudo-Riemannian space is introduced as a carrier of the moving masses, so as to refine and improve the modelling of the description of the free motions of masses by comparing the examples of calculations with experimental data in celestial mechanics.

The main effects are explained by the properties of the Gaussian curvature of pseudo-Riemannian four-dimensional spaces, the weightlessness of moving masses, and the requirements that the orbits of celestial objects should be geodesics, i.e. the accelerations $\mathbf{g}$ (and $\mathbf{a}$) should vanish.

Hence, in the General Theory of Relativity gravitation forces are replaced by the properties of four-dimensional spaces, when each individual mass is specified by constant values of the coordinates ξ^1, ξ^2, ξ^3 and a coordinate proper time τ, which form a system of Lagrange coordinates, always introduced explicitly or implicitly when using mathematical methods of investigations in mechanics, related to construction of theoretical models.

Thus, in the General Theory of Relativity the masses m of individual bodies are constant and non-zero, move with velocities that are constant in value and direction, but in curved space with $\mathbf{g} = \mathbf{a} = 0$. However, in this case, along any orbit of individuals there is no acceleration, and hence the motion of all the masses are inertial, and the corresponding trajectories are said to be geodesics.

Hence, in the General Theory of Relativity all the motions of the masses must occur along geodesic lines when there are no forces, in general, and problems of gravitation theory reduce to determining pseudo-Riemannian spaces filled with geodesic lines.

†*Prikl. Mat. Mekh.* Vol. 60, No. 6, pp. 909–916, 1996.

In this connection, the effects of gravitation forces in the General Theory of Relativity can only arise when models are introduced outside the limits of gravitation theory or one must alter the General Theory of Relativity not only by taking into account the curvature of space, but also introduce the trajectories of the masses with accelerations.

We will assume that the acceleration vectors of the material points **g** and **a** in Eqs (1) are invariant in L under coordinate transformations, when they are determined from the ratio to inertial tetrads or for relative acceleration in comoving coordinates, which are determined by equating to zero the three-dimensional relative velocities $\mathbf{v}_{rel} = 0$ for individuals at rest with respect to moving reference frames. In this case, for individual masses we can only change the coordinate of the proper time τ, but nevertheless the lines L for the corresponding proper time coordinates may have a curvature and, consequently, absolute or relative accelerations.

We can consider the variable coordinate of the proper time of an astronaut, measured by his or her clock, and another clock attached to mass elements, by definition, as synchronized quantities in Newtonian mechanics.

However, in pseudo-Riemannian spaces with a signature $(---+)$ in general global and local time synchronization on different trajectories L for different mass elements is, generally speaking, impossible.

In global comoving systems of coordinates each individual particle, as a consequence of the equality $\mathbf{v}_{rel} = 0$ in the three-dimensional relative sense (the small cabin of a spacecraft with an astronaut fixed in it) can be regarded as being in a state of relative rest.

The acceleration **g** of the gravitation force is generated by attractive forces, according to the law of gravitation, and it is determined by the interactions of masses, while the accelerations **a** are governed by the kinematic properties of the trajectories of the moving elements of the masses considered in four-dimensional spaces.

The actual formulation of mathematical problems and the necessary consequences of the initial equations (1) in gravitation theory are best developed using Lagrange coordinates ξ^1, ξ^2, ξ^3, as components of a vector which retains constant values on L, as which, we can take, in particular, the partial values of certain initial data. It is necessary to introduce such data when integrating the corresponding ordinary differential equations for each separate individual element in the continuous medium considered or for an isolated individual object, undergoing motion in certain fixed pseudo-Riemannian spaces, calculated with respect to local inertial tetrads, considered at points of the lines L.

The phenomenon of weightlessness in a system of freely moving bodies always arises when all the interaction forces for each of the moving elements of the bodies are proportional to their mass.

It is precisely this that is assumed in the gravitation theory, where freely moving systems of bodies and their elements with masses are considered only under the action of modelled overall attractive mass forces $d\mathbf{P} = dm \cdot \mathbf{g}$, and inertial forces $dm \cdot \mathbf{a}$ in the small, according to Eqs (1). The accelerations are determined for each individual particle using comoving holonomically introduced Lagrange coordinates $\xi^1, \xi^2, \xi^3, \tau$ and, in particular, in a system of non-holonomically introduced Fermi coordinates x^1, x^2, x^3. The constant values of the Cartesian-like geometrical coordinates x^1, x^2, x^3 correspond to constant initial data, which are distinguished in four-dimensional spaces together with additional initial-type conditions for the line L—the trajectories of the moving individual point masses, while the time-like variables τ is an indication of the physically defined clocks of the proper time τ attached to them.

As we know, generally speaking, for the Fermi variables x^1, x^2, x^3 introduced, that are non-holonomic by definition, along the lines L as trajectories of the individual elements at all points L, all the Christoffel symbols are zero.

The property of accompaniment in Fermi coordinates for the coordinates of points and space denotes that the contravariant components of the comoving relative three-dimensional velocities $\mathbf{v}_{rel} = 0$ are zero on the lines L. In other words, the following equations hold

$$\mathbf{v}^{\alpha}_{rel} = dx^{\alpha}/d\tau = 0 \text{ and } x^{\alpha} = \text{const on } L \ (\alpha = 1, 2, 3)$$

However, the coordinate transformations $\xi^{\alpha}(x^{\gamma}) = \phi^{\alpha}(x^{\gamma})$, generally speaking, are different for different L. The components $g_{\alpha 4} = u_{\alpha}$ (for metric Riemannian spaces with a metric of the form (3), see below), generally speaking, cannot be made to vanish by a coordinate transformation. In particular, the invariant vector $2\omega = \text{rot}\,\mathbf{u}$ is non-zero, but is constant along L, and a finite body, which has mass, has a constant angular velocity and, correspondingly, a constant angular momentum (we have in mind a mass point with spin) in the gravitation theory.

In general, and particularly in arbitrary pseudo-Riemannian spaces, for the non-holonomically introduced co-moving Fermi variables, the trajectories L_1 and L_2 correspond to different constant values of the coordinates x^1, x^2, x^3 in different Minkowski spaces, respectively shifted translationally or rotated with respect to one another.

Hence, for all spaces in different Fermi variables, representing a Minkowski space, we can put

$$g_{\alpha\beta} dx^\alpha dx^\beta = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 \tag{2}$$

and using additional Lorentz transformations and additional axioms on the invariance of mechanical phenomena for individual points with respect to Lorentz transformations at points L we can obtain a unique Minkowski space (only when (2) holds), as a carrier of events of mechanics in the gravitation theory.

Non-Euclidean four-dimensional Riemannian spaces are used in the gravitation theory and a description of mechanical events, independent of the different reference frames employed, and, in particular, independent of the systems of global comoving Lagrange coordinates, is naturally implied. Here ξ^α are the coordinates of individual points, and an invariant time coordinate of proper time is introduced by means of τ, which requires the introduction of the idea of velocity and acceleration using derivatives with respect to τ.

We will take the metric properties of spaces in the form

$$ds^2 = g_{ij} dz^i dz^j = c^2 d\tau^2 + 2g_{\alpha 4}(\xi^\gamma, \tau) d\xi^\alpha d\tau + g_{\alpha\beta}(\xi^\alpha, \tau) d\xi^\alpha d\xi^\beta, \quad \alpha, \beta, \gamma = 1, 2, 3 \tag{3}$$

In co-moving non-holonomic Fermi systems of coordinates the coordinate lines τ when $\xi^\alpha = \text{const}\ (L)$ correspond to proper time on L for individual masses. We emphasize that the introduction of the space metric in the form (3), where $g_{44} = c^2 = \text{const}$ is immediately possible for all L, where on each line L we have $\xi^1 = \text{const}\ (L)$, $\xi^2 = \text{const}\ (L)$, $\xi^3 = \text{const}\ (L)$, while $ds = c d\tau$ and $ds/d\tau = c = \text{const}$, but the lines L may be different and curved.

In the co-moving metric, for an absolute four-dimensional velocity and the corresponding acceleration along the coordinate line L for proper time τ we can write, by definition

$$\mathbf{u} = d\mathbf{s}/d\tau, \quad \mathbf{a} = d\mathbf{u}/d\tau, \quad u_4 = c$$

where on L

$$\mathbf{u} = c\mathbf{Э}_4 = c\mathbf{e}^4 + cg_{14}\mathbf{e}^1 + cg_{24}\mathbf{e}^2 + cg_{34}\mathbf{e}^3 = u_k \mathbf{e}^k \tag{4}$$

where $\mathbf{Э}_4$ is the unit variable vector, directed along the tangent to L, and the bases $\mathbf{e}^k$ form tetrad contravariant constant inertial bases of local reference frames.

After differentiating (4) with respect to τ we obtain formulae for the components of the absolute accelerations of points with covariant components of the four-dimensional accelerations on L

$$a_\alpha = c\, dg_{\alpha 4}/d\tau, \quad a_4 = 0 \tag{5}$$

Equations (5) of the fundamental and co-moving coordinates also hold on each line L in different non-holonomically introduced Fermi variables locally everywhere along all individual trajectories L in translational and relative motion [1] irrespective of their possible required values for four-dimensional pseudo-Riemannian spaces.

In particular, if a certain scalar $d'U$, with the dimension of specific energy, referred to unit mass in a volume V_4, is a certain infinitely small quantity, which depends, in gravitation theory, only on the coordinates ξ^1, ξ^2, ξ^3 or the Fermi coordinates x^1, x^2, x^3, then obviously $d'U = 0$ on the line L, on which only τ varies, since on L in the co-moving coordinates x^1, x^2, x^3 the quantities ξ^1, ξ^2, ξ^3 are also constant, while $a^4 = 0$ and $v_{\text{rel}} = 0$. (For models outside the framework of the gravitation theory, the specific energy $d'U$ can be variable along L and, consequently, dependent on τ.)

However, the quantity $U(x^\alpha) = \text{const}\ (L)$ may be non-zero along each line L, taking different constant values along different L, and hence, in the co-moving coordinates using (4) we can always write

$$a_\alpha dx^\alpha = c \frac{dg_{\alpha 4}}{d\tau} dx^\alpha = -d'U \tag{6}$$

If we assume that U is a unique function of only the ξ or x coordinates, we can conclude from (5) and (6) that in co-moving coordinates the following conclusion holds

$$cg_{\alpha 4} = u_\alpha = -\frac{\tau}{c}\frac{dU(x^\gamma)}{dx^\alpha} + g_{\alpha 4}(x^\gamma), \quad \alpha, \gamma = 1,2,3; \quad g_{44} = c^2 \tag{7}$$

Correspondingly, for co-moving metrics in certain problems of the Theory of Relativity and, in particular, the gravitation theory for individual points we can use the following formula for the metric

$$ds^2 = c^2 d\tau^2 - 2\frac{\tau}{c} dU d\tau + \frac{\hat{u}_\alpha(\xi^\gamma)}{c} d\xi^\alpha d\tau - dl^2 \tag{8}$$

where $dl^2 = -g'_{\alpha\beta}(x^\gamma, \tau)dx^\alpha dx^\beta$ or $dl^2 = -g'_{\alpha\beta}(\xi^\gamma, \tau)d\xi^\alpha d\xi^\beta$, where, in Fermi variables $dU(\xi^\alpha) = dU(x^\alpha)$ and $x^\alpha = \phi^\alpha(\xi^\gamma)$, but the functions are, generally speaking, different for different L.

It is useful to note the following important fact. Metrics (8) in co-moving coordinates ξ^α, τ and Fermi coordinates x, τ can be rewritten in the following form

$$ds^2 = \left(c\tau - \frac{1}{c^2} dU\right)^2 + \hat{u}_\alpha(\xi^\gamma) d\xi^\alpha d\tau + \hat{g}_{\alpha\beta} d\xi^\alpha d\xi^\beta \tag{9}$$

but $dU(\xi) = dU(x) \neq 0$ (only for shifts with a transfer from one co-moving line L to the neighbouring one do similar increments of the potential energy dU occur and correspondingly equal accelerations in the co-moving systems (8) and (9), which can be regarded as identical four-dimensional and three-dimensional accelerations in orbits L and which, in general, may be non-zero or in both cases equal to zero).

If we put $u_\alpha = 0$ and correspondingly $\omega = 1/2 \text{ rot } \mathbf{u} = 0$, then for different L and correspondingly different values of the global time τ, defined invariantly, we can write metric (8) in the form

$$ds^2 = c^2 d\tau^2 - dl^2 \tag{10}$$

and, in particular, in the Special Theory of Relativity in the form

$$ds^2 = c^2 d\tau^2 - [(dx^1)^2 + (dx^2)^2 + (dx^3)^2] \tag{11}$$

It is obvious that if we specify the accelerations by (5) along all coordinate lines L with proper time τ in non-holonomically introduced Fermi variables on each L, then the coordinates lines L are defined by the Seré–Frenet formula where, irrespective of the space considered, both for absolute and for relative acceleration vectors $a \neq 0$ or $a = 0$.

As we know from the history of the development of relativity theory, the following fundamental invariant equation of field theory in the General Theory of Relativity has been proposed

$$R_i^j - \frac{1}{2}\delta^j R = kT_i^j \tag{12}$$

The left-hand side of this equation is the result of a variation of the expression $R(2k)^{-1}dV_4$ with the dimension of energy, where R is the Gaussian curvature, dV_4 is the invariant element of a four-dimensional spatial volume of Riemann space, orthogonal to the line τ, the constant k with dimension s^2/(cm g) is an extremely small gravitational constant of the General Theory of Relativity, while T_i^j are the components of the energy–momentum tensor, specified for internal points in the specific models investigated.

Equations (12) and relations (13) connected with them have the following common properties.

1. It is possible to obtain detailed solutions in arbitrary pseudo-Riemannian spaces. A sufficient condition for obtaining any solutions in Riemannian spaces is for the Bianchi tensor to vanish, i.e.

$$\nabla_j\left(R_i^j - \frac{1}{2}\delta_i^j R\right) = 0 \tag{13}$$

2. The physical properties of mass interaction, governed by the law of gravitation, are ensured in co-moving Fermi variables x^1, x^2, x^3 with the explicit addition to Eqs (12) of Poisson's equation for the potential energy $U(x^1, x^2, x^3)$

$$\Delta U = -4\pi\rho G \tag{14}$$

It follows from (14) that if this is taken into account, the law of gravitation in co-moving coordinates leads, as a result, to the following important conclusion

$$\rho(x^1, x^2, x^3) = dm/dV_3 \text{ (for } dm = \text{const}, dV_3 = \text{const along the lines)}$$

3. The tensor T_i^j in (12) can have any value, but only in physically justified problems is it best to use the expressions when T_i^j occurs, for the model problem considered, in the equations

$$\nabla_j T_i^j = 0 \tag{15}$$

The fundamental problems in constructing physical-mechanical models consist of an appropriate determination of the components of T_i^j as a function of the corresponding governing parameters.

4. On the other hand, models are possible in which a separate treatment of relations (13) and (15) in constructions of the modelling is impossible, when the global four-dimensional space is not Riemannian. Hence, to obtain physical results, important additions of a different kind will be required.

It should be noted that, when modelling different phenomena, generally speaking, it is not always essential to use the global ideas of space and time defined in the problem considered.

The representation of proper time, introduced non-holonomically in Fermi variables, is now well understood. We will merely add that the Newtonian theory is based on non-holonomic representations of the description of continua in gravitation theories as for different individual masses in separate different spaces. The trajectories of several moving points can be considered in different four-dimensional spaces, and some of their characteristics (for example, the proper times) may agree exactly irrespective of the properties of the different spaces. Hence, the dynamic model equations in gravitation theory can be written in the following forms.

Holonomically in ξ^α, τ variables

$$\nabla_j T_i^j = 0$$

and, as a result of local transformations, non-holonomically in the Fermi variables x^α, τ

$$dT_i^j/dx_j = 0$$

If we take into account the fact that the dynamic equations (14) are true in general, it follows from (10)–(14) that Eqs (12) are not closed, since we obtain from (14) that in non-holonomic form the theory holds for any families of lines L.

Note that the derivation of Eqs (12) using "integrable principles" [2] is essentially based on taking the law of energy conservation into account, which is demonstrated in many textbooks in final form in the General Theory of Relativity.

We can further write the equation

$$-\frac{R}{2k} dV_4 = E dV_3 d\tau \tag{16}$$

where the quantity E has the meaning of the local specific energy, while the coefficient k has the meaning of the gravitational constant, which is related to the Newtonian gravitational constant G and, in the General Theory of Relativity, is assumed to be equal to

$$k = 8\pi c_i/c^1 = 2.07 \times 10^{-48}\, c^2/(\text{cm g}) \tag{17}$$

and is included in al lists of physical constants in physics publications. An independent determination of k by direct experiment is obviously impossible.

It follows from (15) and (17) that the specific curvature of the corresponding Riemannian space is given by the formula

$$R = 4 \times 14 \times 10^{-48}\, E \approx 0 \tag{18}$$

and when constructing models of the gravitation theory can be equated to zero in many applications. However, it is obvious that refinement of the modelling by refining the law of gravitation may turn out to be much more important in descriptions of many natural phenomena.

Basing ourselves on the general principles of the use of modelling and the physical possibilities of macroscopic methods of taking the specific energy E into account, it is natural to assume that in models of the gravitation theory $R = 0$ in ξ^α, τ variables. Hence, we obtain, as a consequence, that Eq. (12) cannot be used to construct macroscopic mechanical models if we wish the modelling of the law of universal mass attraction to be close to reality. In addition it follows from the above that a sensible final value of the potential energy $U(x^1, x^2, x^3)$, which must necessarily be taken into account, cannot be regarded as a replacement of the Gaussian curvature in special Riemannian spaces in gravitation theory.

Note also that in a detailed consideration of the gravitation theory in the General Theory of Relativity it is found that gravity forces due to the geodesic nature of the orbits in the General Theory of Relativity generally do not exist; there is only a curvature of four-dimensional pseudo-Riemannian space due to the fact that $R \neq 0$, since, according to Eq. (12) all the orbits of all individual mass particles in the General Theory of Relativity must be geodesics, as stated in many prestigious textbooks.

However, in nature, the gravity forces described in gravitation theories must occur not only in Newtonian theory in celestial mechanics, but must also cause appropriate accelerations of masses, in particular, resulting from the potential energy, i.e. the presence of the function $U(\xi^\alpha)$ or $U(x^\alpha)$, which is shown to be possible in the Special Theory of Relativity and in Riemannian spaces [3].

In a large number of mechanical problems and events gravity forces manifest themselves in a significant way and can be measured in any motions by special instruments. We mention the possibility of measuring gravity forces (in rocket flights) using instruments attached to moving masses; the gravity force is balanced in the instruments by inertial forces and other internal forces in free flight.

It is obvious that, generally speaking, the replacement of gravity forces by the geometrical properties of spaces in quite unnatural and inappropriate.

Hence, Eqs (1), which are the basis of the equations of weightlessness, enable the above models of gravitation theory to be constructed taking the direct formulation of the law of gravitation into account and possible extensions of the function U, introduced either as a function of the coordinates only in gravitation theory, or as a complete function due to the presence of transforming forms of energy in the mechanical models and problems considered.

However, experiments show that in practice the law of gravitation cannot be disproved! The often-stated assertions that the presence of Gaussian curvature of spaces in gravitation phenomena can automatically replace the action of the attractive forces between masses in a better form is unjustified.

Indeed, as a rule, in the local relationships of many foundations of physical events, which lead to the establishment of differential equations in different mechanical problems, existing gravity forces are not always completely or partially balanced by inertial forces, which are modelled as the source of the properties or geometry of space. Hence, it cannot be asserted that the origin of gravity forces can always be reduced to the geometry of pseudo-Riemannian space. We can list many examples of problems when gravity forces are completely balanced by elastic forces, gas pressure, friction forces, and so on.

Important properties or events in the motions of mechanical systems can in many cases be characterized in an invariant way, by different concepts—not only scalar by also many other invariant concepts, represented, for examples, by vectors and tensors.

Whereas distributed masses in space are represented by non-zero functions only at singular points and lines in empty spaces, the function $U(x^1, x^2, x^3)$ is harmonic and can be represented in terms of specified singularities, where U is a scalar quantity and invariant function, which can be introduced invariantly and always in special local Fermi Cartesian coordinates.

If the scalars mU in co-moving coordinates are constant in coordinates for proper time τ, then, along trajectories L of "point masses at rest" with proper time, the values of the potential energy are constant, but have different values for different point masses.

The co-moving coordinates are coordinates connected with the spacecraft of astronauts or rockets flying freely in space. They can be formed on isolated point masses or for continua in non-holonomically employed Fermi coordinates.

The readings of different instruments, attached to the spacecraft of astronauts "fixed" with respect to the spacecraft and the astronauts, can be reduced to their values for specified observers, including the characteristics and laws of motion of the spacecraft themselves with respect to any specified observers.

The determination of the corresponding laws of motion for the observers is a problem in the theory of inertial navigation [4].

In many publications in gravitation theory only the motions of separate discrete individual interacting masses are considered. In co-moving coordinates the orbits in different Riemannian spaces or the proper time are obtained similarly, but the transformations for transfers between neighbouring orbits in fixed spaces are non-holonomic.

We have considered above motions in the gravitation theory of continuous media, formed by moving masses, on which the conditions of continuity in Riemannian spaces instead of the Newtonian Saint Venant conditions are reduced to constancy of the values of the potential energy in different orbits or the proper time [5] and to the Bianchi identity $\nabla_j(R_i^j - 1/2\delta_i^j R) = 0$

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Translated by R.C.G.